1. (10 points) Give an example of a scalar linear ODE initial value problem which is stable but whose eigenvalue does not always remain below zero. Justify your claim.

2. (a) (5 points) Show that the implicit trapezoidal method is zero-stable for the ODE initial value problem

$$y' = f(t, y), \qquad 0 \leq t \leq b, \tag{1}$$

$$y(0) = c, \tag{2}$$

where f is assumed to be sufficiently smooth and bounded so that the unique existence of a solution is guaranteed with as many bounded derivatives as needed.

(b) (5 points) Prove that the implicit trapezoidal method is convergent of second-order accuracy.

3. (a) (5 points) State the definition of the region of absolute stability for numerical methods of an ODE initial value problem.

   (b) (5 points) Use an example to explain why the absolute stability restriction is a stability, not accuracy, requirement.

4. (a) (5 points) State the definition of A-stability of a numerical method for an ODE initial value problem.

   (b) (5 points) Show that the backward Euler method is A-stable for

$$\begin{aligned} y' &= f(t,y) \qquad 0 \le t \le b, &(3)\\ y(0) &= c. &(4) \end{aligned}$$

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5. Consider a two-step backward differentiation formula (BDF)

$$y_n + \alpha_1 y_{n-1} + \alpha_2 y_{n-2} = h\beta_0 f_n,$$

where $\beta_0 \neq 0$.

(a) (5 points) Determine the unknown coefficients α_1, α_2 and β_0 so that the scheme is second-order accurate.

(b) (5 points) Show that the above method is indeed of second-order accuracy by computing the local truncation error.

6. (10 points) Use the algebraic characterization of stability of BDFs to show that applying the BDF
$$y_n = y_{n-2} + \frac{1}{3}h(f_n + f_{n-1} + f_{n-2})$$
to $y' = \lambda y$ is unstable when $\lambda < 0$.

7. Consider the family of linear multistep methods

$$y_n = \alpha y_{n-1} + \frac{h}{2}(2(1-\alpha)f_n + 3\alpha f_{n-1} - \alpha f_{n-2}),$$

where α is a real parameter.

(a) (10 points) Analyze consistency and order of the methods as functions of α, determining the value α^* for which the resulting method has maximal order.

(b) (10 points) Study the zero-stability of the method with $\alpha = \alpha^*$.

8. (20 points) Formulate the multiple shooting method for the linear problem

$$\mathbf{y}'(t) = A(t)\mathbf{y}(t) + \mathbf{q}(t), \quad 0 \le t \le b, \tag{5}$$
$$B_0\mathbf{y}(0) + B_b\mathbf{y}(b) = \mathbf{b}, \tag{6}$$

where $\mathbf{y}$, $\mathbf{q}$ and $\mathbf{b}$ have m components, and $A(t)$, B_0, and B_b are $m \times m$ matrices.